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Minkowski-space correlators in AdS/CFT correspondence: recipe and applications

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Abstract: We formulate a prescription for computing Minkowski-space correlators from AdS/CFT correspondence. This prescription is shown to give the correct retarded propagators at zero temperature in four dimensions, as well as at finite temperature in the two-dimensional conformal field theory dual to the BTZ black hole. Using the prescription, we calculate the Chern-Simons diffusion constant of the finite-temperature $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the strong coupling limit. We explain why the quasi-normal frequencies of the asymptotically AdS background correspond to the poles of the retarded Green's function of the boundary conformal field theory.

Keywords: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence, Thermal Field Theory.

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1. Introduction

The main prescription of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1], which allows one to compute the correlators in the boundary CFT, was originally formulated for euclidean signature [2, 3], and has been successfully used ever since. Working in euclidean space is a common and convenient practice which usually does not lead to any restrictions since the results obtained in euclidean AdS/CFT can then be analytically continued to Minkowski space, if desired.

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In many cases, however, the ability to extract the lorentzian-signature AdS/CFT results directly from gravity is indispensable. Many interesting properties of gauge theories at finite temperature and density, most notably the response of the thermal ensemble to small perturbations that drive it out of equilibrium, can only be learned from real-time Green's functions. Even in the absence of a gravity dual of QCD, the AdS/CFT correspondence may provide helpful insights into properties of thermal gauge theories at strong coupling.

In principle, one may try to avoid the Minkowski formulation of the AdS/CFT correspondence altogether by working only with the euclidean version and using analytic properties of the Green's functions to find the real-time propagators. In practice, however, such a method is not useful, because analytic continuation to the Minkowski space is possible only when the euclidean correlators are exactly known for *all* Matsubara frequencies. Since gravity calculations in non-extremal backgrounds usually involve approximations of some sort (such as the high- and low-temperature expansions used in ref. [4]), exact expressions involving all Matsubara frequencies are normally beyond reach. It is therefore crucial to have a prescription allowing one to compute the Minkowski correlators directly from gravity.

Subtleties of the lorentzian-signature AdS/CFT correspondence are well known [5, 6, 7]. In this paper we shall not try to give a formulation of the Minkowski AdS/CFT in a form as general as the euclidean version. Our goal is to formulate a working recipe, whose justification and full understanding would hopefully emerge in future research. The simple prescription that will be given is sufficient for the computation of the retarded propagator, and hence all other two-point correlators, but does not allow one to calculate higher-point correlation functions. We defer the treatment of higher-point correlators to future work.

The paper is structured as follows. In section 2 we review the properties of the euclidean and Minkowski thermal Green's functions. We formulate our recipe in section 3 and illustrate it on a simplest zero-temperature example. We compute the Minkowski-space correlators of the operator dual to the dilaton in AdS_5, and show that they agree with the analytic continuation of the classic euclidean result. In section 4 another non-trivial check of the prescription is performed: we compute the retarded Green's functions in the two-dimensional CFT dual to the Bañados-Teitelboim-Zanelli (BTZ) black hole background. We show that the result coincides with the analytic continuation from euclidean space. We also find that the poles of the retarded Green's functions correspond precisely to the quasinormal frequencies of the general BTZ black hole, and we explain why this is to be expected in general in the framework of our recipe. Section 5 is devoted the computation of the Chern-Simons diffusion rate in the the strongly coupled $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory. The result cannot be tested independently, and is considered as a prediction. Section 6 contains the conclusions and outlook. The appendices are devoted to various calculations outside the main line of the paper.

The prescription developed in this paper is further applied to the hydrodynamic regime of the $\mathcal{N}=4$ SYM theory in ref. [10].

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2. Minkowski thermal correlators

Since the main topics of this paper is the computation of thermal Green's functions from gravity, let us review the definition and general properties of different thermal Green's functions. Consider a quantum field theory at finite temperature. Let $\hat{\mathcal{O}}$ be an arbitrary local operator, which we assume, for definiteness, to be bosonic (the fermionic case is completely analogous). In Minkowski space, the retarded propagator for $\hat{\mathcal{O}}$ is defined by

$$G^R(k) = -i \int d^4x \, e^{-ik\cdot x} \, \theta(t) \, \langle [\hat{\mathcal{O}}(x), \hat{\mathcal{O}}(0)] \rangle \tag{2.1}$$

(we use the $--+++$ metric convention, and $t \equiv x^0$). The advanced propagator is defined similarly,

$$G^A(k) = i \int d^4x \, e^{-ik\cdot x} \, \theta(-t) \, \langle [\hat{\mathcal{O}}(x), \hat{\mathcal{O}}(0)] \rangle \, . \tag{2.2}$$

From the definitions can see that $G^R(k)^* = G^R(-k) = G^A(k)$. If the system is P-invariant, then $\operatorname{Re} G^{R,A}$ are even functions of $\omega \equiv k^0$ and $\operatorname{Im} G^{R,A}$ are odd functions of ω. Instead of G^R and G^A, one can consider other correlation functions of $\hat{\mathcal{O}}$ in thermal equilibrium. One example is the symmetrized Wightman function,

$$G(k) = \frac{1}{2} \int d^4x \, e^{-ik\cdot x} \langle \hat{\mathcal{O}}(x)\hat{\mathcal{O}}(0) + \hat{\mathcal{O}}(0)\hat{\mathcal{O}}(x) \rangle \, . \tag{2.3}$$

All other correlation functions can be expressed via G^R, G^A and G. For example, the Feynman propagator is

$$G^F(k) = -i \int d^4x \, e^{-ik\cdot x} \, \langle T\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(0) \rangle = \frac{1}{2}[G^R(k) + G^A(k)] - iG(k) \, . \tag{2.4}$$

From the spectral representations of G^R and G, one can relate $G(k)$ with the imaginary part of the retarded propagator,

$$G(k) = -\coth\frac{\omega}{2T} \operatorname{Im} G^R(k) \, . \tag{2.5}$$

Therefore, if $G^R(k)$ is known, all other correlators are easy to compute. In particular,

$$G^F(k) = \operatorname{Re} G^R(k) + i \coth\frac{\omega}{2T} \operatorname{Im} G^R(k) \, . \tag{2.6}$$

At zero temperature ($T \to 0$), eq. (2.6) reduces to a simple formula,

$$G^F(k) = \operatorname{Re} G^R(k) + i \operatorname{sgn}\omega \operatorname{Im} G^R(k) \, , \qquad T = 0 \, . \tag{2.7}$$

Taking the limit $\omega \to 0$ in eq. (2.5), one obtains another useful formula,

$$G(0, \boldsymbol{k}) = -\lim_{\omega\to 0} \frac{2T}{\omega} \operatorname{Im} G^R(k) = 2iT \frac{\partial}{\partial\omega} G^R(\omega, \boldsymbol{k}) \Big|_{\omega=0} \, . \tag{2.8}$$

The multitude of Minkowski-space Green's functions is in contrast with the situation in euclidean space, where one normally deals only with the Matsubara propagator,

$$G^E(k_E) = \int d^4x_E \, e^{-ik_E\cdot x_E} \langle T_E \hat{\mathcal{O}}(x_E)\hat{\mathcal{O}}(0) \rangle \, . \tag{2.9}$$

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Here T_E denotes euclidean time ordering. The Matsubara propagator is defined only at discrete values of the frequency ω_E. For bosonic $\hat{\mathcal{O}}$ these Matsubara frequencies are multiples of $2\pi T$.

The euclidean and Minkowski propagators are closely related. The retarded propagator $G^R(k)$, as a function of ω, can be analytically continued to the whole upper half plane and, moreover, at complex values of ω equal to $2\pi i T n$, reduces to the euclidean propagator,

$$G^R(2\pi i T n, \boldsymbol{k}) = -G^E(2\pi T n, \boldsymbol{k})\,. \tag{2.10}$$

Analogously, the advanced propagator, analytically continued to the lower half plane, is equal to the Matsubara propagator at the points $\omega = -2\pi i T n$,

$$G^A(-2\pi i T n, \boldsymbol{k}) = -G^E(-2\pi T n, \boldsymbol{k})\,. \tag{2.11}$$

In particular, by putting $n = 0$, one finds that $G^R(0, \boldsymbol{k}) = G^A(0, \boldsymbol{k}) = -G^E(0, \boldsymbol{k})$.

3. Minkowski AdS/CFT prescription

3.1 Difficulties with Minkowski AdS/CFT

Let us first recall the formulation of the AdS/CFT correspondence in euclidean space. For definiteness, we shall talk about the correspondence between the strongly coupled $\mathcal{N} = 4$ SYM theory and classical (super)gravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The euclidean version of the metric of the latter has the form

$$ds^2 = \frac{R^2}{z^2}(d\tau^2 + d\boldsymbol{x}^2 + dz^2) + R^2 d\Omega_5^2\,, \tag{3.1}$$

which is a solution to the Einstein equations. In the AdS/CFT correspondence, the four-dimensional quantum field theory lives on the boundary of the AdS_5 space at $z = 0$. Suppose that a bulk field ϕ is coupled to an operator $\hat{\mathcal{O}}$ on the boundary in such a way that the interaction lagrangian is $\phi\,\hat{\mathcal{O}}$. In this case, the AdS/CFT correspondence is formally stated as the equality

$$\langle e^{\int_{\partial M} \phi_0 \hat{\mathcal{O}}} \rangle = e^{-S_{\rm cl}[\phi]}\,, \tag{3.2}$$

where the left-hand side is the generating functional for correlators of $\hat{\mathcal{O}}$ in the boundary field theory (i.e., $\mathcal{N} = 4$ SU(N) SYM theory at large N and large 't Hooft coupling g^2N), and the exponent on the right-hand side is the action of the classical solution to the equation of motion for ϕ in the bulk metric with the boundary condition $\phi|_{z=0} = \phi_0$.

The metric (3.1) corresponds to the zero-temperature field theory. To compute the Matsubara correlator at finite temperature, one has to replace it by a non-extremal one,

$$ds^2 = \frac{R^2}{z^2}\left(f(z)d\tau^2 + d\boldsymbol{x}^2 + \frac{dz^2}{f(z)}\right) + R^2 d\Omega_5^2\,, \tag{3.3}$$

where $f(z) = 1 - z^4/z_H^4$ and $z_H = (\pi T)^{-1}$, and T is the Hawking temperature. The euclidean time coordinate τ is periodic, $\tau \sim \tau + T^{-1}$, and z runs between 0 and z_H.

One can try to formally write the Minkowski version of the AdS/CFT as the equivalence

$$\langle e^{i\int_{\partial M}\phi_0\mathcal{O}}\rangle = e^{iS_{\rm cl}[\phi]}\,. \tag{3.4}$$

However, an immediate problem arises. In the euclidean version, the classical solution ϕ is uniquely determined by its value ϕ_0 at the boundary $z=0$ and the requirement of regularity at the horizon $z=z_H$. Correspondingly, the euclidean correlator obtained by using the correspondence is unique. In contrast, in the Minkowski space, the requirement of regularity at the horizon is insufficient; to select a solution one needs a more refined boundary condition there. This problem is well known [5], and is thought to reflect the multitude of real-time Green's functions (Feynman, retarded, advanced) in finite-temperature field theory. Previous discussion of the lorentzian signature AdS/CFT appears also in refs. [6]–[9].

One boundary condition at the horizon stands out from the physical point of view. It is the incoming-wave boundary condition, where waves can only travel to the region inside the horizon of the black branes, but cannot be emitted from there. We may suspect that this boundary condition corresponds to the retarded Green's function, while the outgoing-wave boundary condition gives rise to the advanced Green's function. However, even after fixing the boundary condition at the horizon, it is still problematic to get eq. (3.4) to work.

To see where the problem lies, first we notice that the AdS part of the metric (3.3) is of the form

$$ds^2 = g_{zz}dz^2 + g_{\mu\nu}(z)dx^\mu dx^\nu\,. \tag{3.5}$$

Let us consider fluctuations of a scalar (e.g., the dilaton or the axion) on this background. The action of the scalar reads

$$S = K\int d^4x \int_{z_B}^{z_H} dz\,\sqrt{-g}\left[g^{zz}(\partial_z\phi)^2 + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]\,, \tag{3.6}$$

where K is a normalization constant (for the dilaton $K=-\pi^3R^5/4\kappa_{10}^2$, κ_{10} is the ten-dimensional gravitational constant) and m is the scalar mass. The limit of the z-integration is between the boundary z_B and the horizon z_H (for the metric (3.3), $z_B=0$).

The linearized field equation for ϕ,

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}\,g^{zz}\partial_z\phi) + g^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0\,, \tag{3.7}$$

has to be solved with fixed boundary condition at z_B. The solution is

$$\phi(z,x) = \int\frac{d^4k}{(2\pi)^4}\,e^{ik\cdot x}f_k(z)\phi_0(k)\,, \tag{3.8}$$

where $\phi_0(k)$ is determined by the boundary condition,

$$\phi(z_B,x) = \int\frac{d^4k}{(2\pi)^4}\,e^{ik\cdot x}\phi_0(k)\,, \tag{3.9}$$

and $f_k(z)$ is the solution to the mode equation,

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}\,g^{zz}\partial_z f_k) - (g^{\mu\nu}k_\mu k_\nu + m^2)f_k = 0\,, \tag{3.10}$$

with unit boundary value at the boundary, $f_k(z_B)=1$ and satisfying the incoming-wave boundary condition at $z=z_H$.

The action on shell (i.e., for classical solutions) reduces to the surface terms

$$S = \int \frac{d^4k}{(2\pi)^4}\, \phi_0(-k)\mathcal{F}(k,z)\phi_0(k)\Big|_{z=z_B}^{z=z_H}\,, \tag{3.11}$$

where

$$\mathcal{F}(k,z) = K\sqrt{-g}g^{zz}f_{-k}(z)\partial_z f_k(z)\,. \tag{3.12}$$

Now, if one try to make the identification (3.4), the Green function can be obtained by taking second derivative of the classical action with respect to the boundary value of ϕ. From eq. (3.11), this Green's function is

$$G(k) = -\mathcal{F}(k,z)\Big|_{z_B}^{z_H} - \mathcal{F}(-k,z)\Big|_{z_B}^{z_H}\,. \tag{3.13}$$

One can see, however, that this quantity is completely real, and hence cannot be a candidate to the retarded Green's function, which is in general complex. Indeed, by noticing that $f_k^*(z) = f_{-k}(z)$ (this is because $f_k^*(z)$ is also a solution to eq. (3.10) and also satisfies the incoming-wave boundary condition), one sees that the imaginary part of $\mathcal{F}(k,z)$ is proportional a conserved flux,

$$\mathrm{Im}\,\mathcal{F}(k,z) = \frac{K}{2i}\sqrt{-g}\,g^{zz}\,[f_k^*\partial_z f_k - f_k\partial_z f_k^*]\,, \tag{3.14}$$

and thus $\mathrm{Im}\,\mathcal{F}(k,z)$ is independent of the radial coordinate, $\partial_z\,\mathrm{Im}\,\mathcal{F} = 0$. Therefore, in each term in the right-hand side of eq. (3.13), the imaginary part at the horizon $z = z_H$ completely cancels out the imaginary part at the boundary $z = z_B$.

We should clarify one issue with the boundary term at the horizon. It is commonly thought that this term is oscillating and hence averaged out to zero. However, this is not the case if the incoming-wave boundary condition is consistently maintained. We shall see that explicitly below in section 3.3.

We can try to avoid this problem by throwing away the contribution from the horizon, keeping only the boundary term at $z = z_B$. However, now the imaginary parts of the two terms in eq. (3.13) cancel each other: from the reality of the field equation one can show that $\mathcal{F}(-k,z) = \mathcal{F}^*(k,z)$. Therefore, the resulting $G(k)$ is still real. It seems that by differentiating the classical action one cannot get the retarded Green's function, which is, in general, complex.

3.2 The Minkowski prescription

We circumvent the difficulties mentioned above by putting forward the following conjecture

$$G^R(k) = -2\mathcal{F}(k,z)\Big|_{z_B}\,. \tag{3.15}$$

which seems rather natural but, for reasons explained above, does not follow strictly from an identity of the type (3.4). The justification for eq. (3.15) is that it works in all cases where independent verification is possible.

Our prescription for computing the retarded (advanced) Green's functions is thus formulated as follows:

1. Find a solution to the mode equation (3.10) with the following properties:
   - The solution equals to 1 at the boundary $z = z_B$;
   - For timelike momenta, the solution has an asymptotic expression corresponding to the incoming (outgoing) wave at the horizon. For spacelike momenta, the solution is regular at the horizon.
2. The retarded Green's function is then given by $G = -2\mathcal{F}_{\partial M}$, where $\mathcal{F}$ is given in eq. (3.12). Only the contribution from the boundary has to be taken. Surface terms coming from the horizon or, more generally, from the "infrared" part of the background geometry (corresponding to the "position of the branes") must be dropped. This part of the metric influences the correlators only through the boundary condition imposed on the bulk field ϕ.

Since the imaginary part of $\mathcal{F}(k,z)$ is independent of the radial coordinate, $\operatorname{Im} G^{R,A}$ can be computed by evaluating $\operatorname{Im}\mathcal{F}(k,z)$ at any convenient value of z. In particular, it can be computed at the horizon.

In order to verify that the prescription indeed gives the retarded propagator G^R, one should compute G^R in theories where it is known from other methods. Below we perform a check for zero-temperature field theory; further checks are done in section (4) and ref. [10].

3.3 Example: zero-temperature field theory

As an example, we use the prescription (3.15) to compute the retarded (advanced) two-point function of the composite operators $\mathcal{O} = \frac{1}{4}F^2$ at zero temperature. In this case the action (3.6) is that for a minimally coupled massless scalar on the background (3.1). The horizon is at $z_H = \infty$ and the boundary at $z_B = \epsilon \to 0$. The mode equation reads

$$f_k''(z) - \frac{3}{z} f_k'(z) - k^2 f_k(z) = 0\,. \tag{3.16}$$

For spacelike momenta, $k^2 > 0$, the calculation is identical to the one for the euclidean case [2] (see also appendix A), the only difference being the extra minus sign in front of the lorentzian signature action. We obtain

$$G_R(k) = \frac{N^2 k^4}{64\pi^2} \ln k^2\,, \qquad k^2 > 0\,. \tag{3.17}$$

For timelike momenta, we introduce $q = \sqrt{-k^2}$. The solution to eq. (3.16) satisfying boundary conditions outlined above is given by

$$f_k(z) = \begin{cases} \dfrac{z^2 H_2^{(1)}(qz)}{\epsilon^2 H_\nu^{(1)}(q\epsilon)}\,, & \omega > 0, \\[2ex] \dfrac{z^2 H_2^{(2)}(qz)}{\epsilon^2 H_2^{(2)}(q\epsilon)}\,, & \omega < 0. \end{cases} \tag{3.18}$$

Notice that $f_{-k} = f_k^*$. Computing (3.12) and using eq. (3.15), we get

$$G^R(k) = \frac{N^2 k^4}{64\pi^2}(\ln k^2 - i\pi \,\mathrm{sgn}\, \omega)\,. \tag{3.19}$$

Combining eqs. (3.17) and (3.19), we obtain the retarded Green's function,

$$G^R(k) = \frac{N^2 k^4}{64\pi^2}(\ln |k^2| - i\pi \,\theta(-k^2)\,\mathrm{sgn}\, \omega)\,. \tag{3.20}$$

Using the asymptotics of the Hankel functions for $z \to \infty$, one can check that $\mathcal{F}(k,z)$ is not zero at large $z \to \infty$. Moreover, it is purely imaginary in this limit, and is equal to $iN^2k^4\,\mathrm{sgn}\,\omega/128\pi = \mathrm{Im}\,\mathcal{F}(k,\epsilon)$, as it should be in view of the flux conservation (3.14). Thus, the imaginary part of the Green's function can be computed independently using the asymptotics of the solution at the horizon.

By using the relation between the Feynman and retarded propagators at zero temperature, eq. (2.7), we find

$$G^F(k) = \frac{N^2 k^4}{64\pi^2}(\ln |k^2| - i\pi \,\theta(-k^2))\,, \tag{3.21}$$

which can be obtained from the euclidean correlator,

$$G_E(k_E) = -\frac{N^2 k_E^4}{64\pi^2}\ln k_E^2\,, \tag{3.22}$$

by a Wick rotation. Thus, our prescription reproduces the correct answer for the retarded Green's function at zero temperature.

The euclidean and Minkowski correlators corresponding to massive scalar modes in AdS_5 are computed in appendix A.

4. Retarded correlators in 2d CFT and the BTZ black hole

In this section, we use the prescription (3.15) in the general BTZ black hole background to compute the retarded Green's functions of the dual two-dimensional CFT. The result is then analytically continued to complex frequencies, and is shown to reproduce the Matsubara correlators known from field theory. The calculation is technically quite involved, but is nevertheless instructive since it is the first nontrivial check of our prescription at finite temperature. As a by-product, we demonstrate that the gravitational quasinormal frequencies in the asymptotically AdS background correspond to the poles of the retarded correlator in the dual CFT on the boundary, and thus provide a quantitative description of the system's return to thermal equilibrium.

4.1 Gravity calculations

The non-extremal BTZ black hole [11, 12] is a solution of the $2+1$-dimensional Einstein equations with a cosmological constant $\Lambda = -1/l^2$. The metric is given by

$$ds^2 = -\frac{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}{l^2\rho^2}dt^2 + \rho^2\left(d\varphi - \frac{\rho_+\rho_-}{l\rho^2}dt\right)^2 + \frac{l^2\rho^2}{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}d\rho^2\,, \tag{4.1}$$

where φ is periodic with the period 2π and $\rho_\pm$ correspond to the positions of the inner/outer horizons. Introducing new variables μ, $x_\pm$, with

$$\begin{aligned} \rho^2 &= \rho_+^2 \cosh^2\mu - \rho_-^2 \sinh^2\mu\,, \\ x^\pm &= \pm\frac{\rho_\pm t}{l} \mp \rho_\mp \varphi\,, \end{aligned} \tag{4.2}$$

we can write the metric as

$$ds^2 = l^2 d\mu^2 - \sinh^2\mu (dx^+)^2 + \cosh^2\mu (dx^-)^2\,. \tag{4.3}$$

In terms of $z = \tanh^2\mu$, the action of a massive scalar in the BTZ background reads

$$\begin{aligned} S = -\,\eta \int dz\, dx^+\, dx^- \, \frac{l}{2(1-z)^2}\Bigg[&\frac{4z(1-z)^2}{l^2}\,(\partial_z\phi)^2 - \frac{1-z}{z}\,(\partial_+\phi)^2 + \\ &+ (1-z)\,(\partial_-\phi)^2 + m^2\phi^2\Bigg]\,, \end{aligned} \tag{4.4}$$

where η is the normalization constant.

The mode functions $f_{k_+,k_-}(z)$ are defined so that

$$\phi(x^\pm, z) = \int \frac{dk_+\, dk_-}{(2\pi)^2}\, e^{-i(k_+x^+ + k_-x^-)} f_{k_+,k_-}(z)\phi_0(k_+,k_-)\,, \tag{4.5}$$

and satisfy the equation

$$\partial_{zz}^2 f_{k_+,k_-} + \frac{1}{z}\partial_z f_{k_+,k_-} + \left[\frac{l^2k_+^2}{4z^2(1-z)} - \frac{l^2k_-^2}{4z(1-z)} - \frac{m^2l^2}{4z(1-z)^2}\right] f_{k_+,k_-} = 0\,. \tag{4.6}$$

On shell, the action (4.4) reduces to the boundary terms,

$$S = \int \frac{dk_+dk_-}{(2\pi)^2}\,\phi_0(-k_+,-k_-)\left[\mathcal{F}(k_+,k_-,z_B) - \mathcal{F}(k_+,k_-,z_H)\right]\phi_0(k_+,k_-)\,, \tag{4.7}$$

where $z_B = 1$ corresponds to the boundary at infinity ($\rho = \infty$), $z_H = 0$ is the location of the outer horizon ($\rho = \rho_+$). The function $\mathcal{F}$ is given by

$$\mathcal{F}(k_+,k_-,z) = -\eta\frac{2z}{l} f_{k_+,k_-}^*\partial_z f_{k_+,k_-}\,. \tag{4.8}$$

According to the Minkowski AdS/CFT prescription, the retarded/advanced Green's functions are determined by

$$G^{R,A}(k_+,k_-) = -2[\lim_{z\to z_B}\mathcal{F}(k_+,k_-,z)]\,, \tag{4.9}$$

where the solution $f_{k_+,k_-}(z)$ is normalized to 1 at $z = 1-\epsilon$ and represents a purely incoming (outgoing) wave at the horizon $z = 0$. Square brackets indicate that the contact terms are ignored. The imaginary part of $G^{R,A}$, due to the flux conservation, can be computed either at the boundary or at the horizon,

$$\operatorname{Im} G^{R,A}(k_+,k_-) = -2[\lim_{z\to z_B}\operatorname{Im}\mathcal{F}(k_+,k_-,z)] = -2[\lim_{z\to z_H}\operatorname{Im}\mathcal{F}(k_+,k_-,z)]\,. \tag{4.10}$$

The solution to eq. (4.6) can be written as

$$f_{k_+,k_-}(z) = z^\alpha (1-z)^\beta \epsilon^{-\beta}(1-\epsilon)^{-\alpha} \frac{{}_2F_1\,(a,b;c;z)}{{}_2F_1\,(a,b;c;1-\epsilon)}\,, \tag{4.11}$$

where

$$a,b = \frac{l(k_+ \mp k_-)}{2i} + \beta\,, \qquad c = 1 + 2\alpha\,, \tag{4.12}$$

and the indices α, β are given by

$$\alpha_\pm = \pm i l k_+/2\,, \qquad \beta_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 + l^2 m^2}\right) = \frac{\Delta_\pm}{2}\,, \tag{4.13}$$

α_- corresponds to the incoming wave at the horizon, α_+ to the outgoing one, and $\Delta_\pm$ is the conformal weight of the boundary CFT operator. The range of Δ_+ is $[1,\infty)$, while the range of Δ_- is $(0,1]$, where zero corresponds to the unitarity bound in $d=3$. Here we shall consider only the Δ_+ branch. Accordingly,[1] we choose $\beta = \beta_-$ in eq. (4.11).

4.1.1 The full retarded Green's function

Near the boundary, i.e., at $z = 1-\epsilon$, the derivative of the solution (4.11) takes the form

$$f'_{k_+,k_-}(z)|_{z=1-\epsilon} = -\frac{\beta}{\epsilon} + \frac{ab\,{}_2F_1\,(a+1,b+1;c+1;1-\epsilon)}{c\,{}_2F_1\,(a,b;c;1-\epsilon)}\,, \tag{4.14}$$

The retarded Green's function is then computed using the prescription (4.9),

$$G^R(k_+,k_-) = \frac{4\eta}{l}\left[\frac{ab}{c}\lim_{\epsilon\to 0}\frac{{}_2F_1\,(a+1,b+1;c+1;1-\epsilon)}{{}_2F_1\,(a,b;c;1-\epsilon)}\right]\,. \tag{4.15}$$

Some care should be taken when computing the limit in (4.15) for $1-2\beta_+ = c-a-b = -n$, with $n = 0,1,2,\ldots$ i.e., for $\Delta_+ = 2\beta_+ \neq n+1$. In this case the hypergeometric function is degenerate and contains logarithmic terms. Accordingly, integer and non-integer Δ_+ need to be considered separately.

4.1.2 Non-integer Δ_+

For $\Delta_+ = 2\beta_+ \neq n+1$, where $n = 0,1,2,\ldots$ direct calculation gives

$$\begin{aligned} G_R(p_+,p_-) = &-\frac{2\,\eta\,\epsilon^{2\beta_+-2}}{\pi\,l\,\Gamma^2(2\beta_+-1)\sin 2\pi\beta_+}\left|\Gamma\left(\beta_+ + \frac{ip_+}{2\pi T_L}\right)\Gamma\left(\beta_+ + \frac{ip_-}{2\pi T_R}\right)\right|^2 \times \\ &\times\left[\cosh\left(\frac{p_+}{2T_L} - \frac{p_-}{2T_R}\right) - \cos 2\pi\beta_+ \cosh\left(\frac{p_+}{2T_L} + \frac{p_-}{2T_R}\right) + \right. \\ &\left. + i\sin 2\pi\beta_+ \sinh\left(\frac{p_+}{2T_L} + \frac{p_-}{2T_R}\right)\right]\,, \end{aligned} \tag{4.16}$$

where $T_{L,R} = (\rho_+ \mp \rho_-)/2\pi$, and the momenta $p_\pm$ are related to $k_\pm$ and ω, k as follows

$$p_+ = \pi T_L\,(k_+ + k_-)\,l = \frac{\omega - k}{2}\,, \qquad p_- = \pi T_R\,(k_+ - k_-)\,l = \frac{\omega + k}{2}\,. \tag{4.17}$$

[1] In AdS/CFT, if a supergravity scalar behaves near the boundary as $f \sim A\epsilon^{d-\Delta_+} + B\epsilon^{\Delta_+}$, A is interpreted as the source of a dimension-Δ_+ operator.

4.1.3 Integer Δ_+

In the case of integer conformal dimension, one has to further distinguish between Δ_+ being an odd or even integer. In the latter case, i.e., for $h_L = h_R = \beta_+ = \Delta_+/2 = n+1$, $n = 0, 1, 2, \ldots$, we get

$$\begin{aligned} G_R(p_+, p_-) =& \frac{4\eta\epsilon^{2n}}{l\pi^2\Gamma^2(2n+1)} \left|\Gamma\left(1+n+\frac{ip_+}{2\pi T_L}\right)\Gamma\left(1+n+\frac{ip_-}{2\pi T_R}\right)\right|^2 \times \\ & \times \sinh\frac{p_+}{2T_L}\sinh\frac{p_-}{2T_R}\left[\psi\left(1+n-\frac{ip_+}{2\pi T_L}\right)+\psi\left(1+n-\frac{ip_-}{2\pi T_R}\right)\right]. \end{aligned} \tag{4.18}$$

In particular, the following simple result is obtained for $h_L = h_R = 1$ ($\Delta_+ = 2$):

$$G_R(p_+, p_-) = \frac{\eta p_+ p_-}{\pi^2 l T_L T_R}\left[\psi\left(1-\frac{ip_+}{2\pi T_L}\right)+\psi\left(1-\frac{ip_-}{2\pi T_R}\right)\right]. \tag{4.19}$$

The imaginary part of (4.19),

$$\operatorname{Im} G_R(p_+, p_-) = -\frac{\eta p_+ p_-}{2\pi l T_L T_R}\left(\coth\frac{p_+}{2T_L}+\coth\frac{p_-}{2T_R}\right), \tag{4.20}$$

coincides with the appropriate limit of eq. (4.24) below. For odd integer values of Δ_+, i.e., for $h_L = h_R = \beta_+ = \Delta_+/2 = n + 1/2$, $n = 1, 2, \ldots$, we get

$$\begin{aligned} G_R(p_+, p_-) =& \frac{4\eta\epsilon^{2n-1}}{l\pi^2\Gamma^2(2n)} \left|\Gamma\left(\frac{1}{2}+n+\frac{ip_+}{2\pi T_L}\right)\Gamma\left(\frac{1}{2}+n+\frac{ip_-}{2\pi T_R}\right)\right|^2 \times \\ & \times \cosh\frac{p_+}{2T_L}\cosh\frac{p_-}{2T_R}\left[\psi\left(\frac{1}{2}+n-\frac{ip_+}{2\pi T_L}\right)+\psi\left(\frac{1}{2}+n-\frac{ip_-}{2\pi T_R}\right)\right]. \end{aligned} \tag{4.21}$$

Finally, for $\beta_+ = 1/2$ ($\Delta_+ = 1$) we have

$$G_R(p_+, p_-) = \frac{4\,\eta}{l\epsilon\ln^2\epsilon}\left[\psi\left(\frac{1}{2}-\frac{ip_+}{2\pi T_L}\right)+\psi\left(\frac{1}{2}-\frac{ip_-}{2\pi T_R}\right)\right]. \tag{4.22}$$

The imaginary part of the obtained Green's functions coincides with eq. (4.24) below. More crucially, one can check that the only singularities of (4.16), (4.18), (4.21) and (4.22) are simple poles located in the lower half-plane of complex ω at

$$\omega_n^{(L)} = k - i4\pi T_L(h_L + n)\,, \qquad n = 0, 1, 2, \ldots \tag{4.23a}$$

$$\omega_n^{(R)} = -k - i4\pi T_R(h_R + n)\,, \qquad n = 0, 1, 2, \ldots \tag{4.23b}$$

No singularity is located in the upper half plane, as expected for the retarded Green's function.

4.1.4 A note about the imaginary part

Notice that if we are interested only in the imaginary part of the retarded propagator, the calculations are much simpler because it can be done by using the solution (4.11) with $\alpha = \alpha_-$, $\beta = \beta_-$ and taking the limit $z \to z_H$ in eq. (4.10). The result reads

$$\operatorname{Im} G_R = -\,\mathcal{C}(\epsilon, h_{L,R})\sinh\left(\frac{p_+}{2T_L}+\frac{p_-}{2T_R}\right)\left|\Gamma\left(h_L+\frac{ip_+}{2\pi T_L}\right)\Gamma\left(h_R+\frac{ip_-}{2\pi T_R}\right)\right|^2, \tag{4.24}$$

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where $h_L = h_R = \beta_+$. The normalization constant is

$$\mathcal{C}(\epsilon, h_{L,R}) = \begin{cases} \dfrac{2\,\eta\,\epsilon^{h_L+h_R-2}}{\pi\, l\, \Gamma(2h_L-1)\Gamma(2h_R-1)}\,, & \Delta_+ > 1\,, \\ \dfrac{2\,\eta}{\pi\, l\, \epsilon \ln^2 \epsilon}\,, & \Delta_+ = 1\,. \end{cases} \tag{4.25}$$

This result has been known before from the absorption calculations [13].

4.2 Comparison with the two-dimensional finite-temperature CFT

Now let us show that the retarded Green's function found from gravity coincides with the field-theory result. One starts from the coordinate-space expression for the euclidean finite-temperature correlation function of the local operator $\mathcal{O}$ with conformal dimensions $(h, \bar{h})$ [14]

$$G^E(\tau, x) = \langle \mathcal{O}(w, \bar{w}) \mathcal{O}(0,0) \rangle = \frac{C_{\mathcal{O}}\, (\pi T)^{4h+4\bar{h}}}{\sinh^{2h}[\pi T w]\, \sinh^{2\bar{h}}[\pi T \bar{w}]}\,, \tag{4.26}$$

where $C_{\mathcal{O}}$ is a normalization constant and $w = x + i\tau$, with x running from $-\infty$ to $+\infty$, and τ being periodic with the period $1/T$.

To compare with the results obtained from gravity, we need to compute the Fourier transform of (4.26) for $h = \bar{h}$, obtain the momentum-space Matsubara correlator, and compare the result with the retarded Green's functions obtained from gravity in section 4.1.1. We make the comparison only for $T_L = T_R$ and for the integer values of $\Delta = 2h = 2\bar{h}$. We need to compute the following integral,

$$G^E(\omega_E = 2\pi n T, k) = \int_0^{1/T} d\tau \int dx\, e^{i\omega_E \tau}\, e^{ikx}\, G^E(\tau, x)\,. \tag{4.27}$$

It is convenient to write $G^E(\tau, x)$ in the form

$$G^E(\tau, x) = \frac{C_{\mathcal{O}} (2\pi T)^{2\Delta}}{(\cosh 2\pi T r - \cos 2\pi T \tau)^{\Delta}}\,. \tag{4.28}$$

We shall first take the integral over τ and then over x. Introducing $z = \exp(-i 2\pi T \tau)$ and $r = |x|$, the integral becomes

$$G^E(\omega_E, k) = \frac{(-1)^{2\Delta} 2 (2\pi T)^{2\Delta} C_{\mathcal{O}}}{T} \int_0^{\infty} dr \cos kr \oint_{|z|=1} \frac{dz}{2\pi i} \frac{z^{\Delta-n-1}}{(z - z_+)^{\Delta} (z - z_-)^{\Delta}}\,, \tag{4.29}$$

where $z_{\pm} = \exp(\pm 2\pi T r)$. The integral over z is computed by evaluating the residues. The result reads

$$G^E(\omega_E, k) = \frac{2 (2\pi T)^{2\Delta} C_{\mathcal{O}}}{T} \frac{\Gamma(\Delta - n)}{\Gamma(\Delta)} \sum_{m=0}^{\Delta-1} \frac{\Gamma(2\Delta - m - 1)}{m! \Gamma(\Delta - m) \Gamma(\Delta - m - n)} J_{n,m}(k)\,, \tag{4.30}$$

where $J_{n,m}(k)$ are the following integrals that can be computed explicitly [15],

$$J_{n,m}(k) = \int_0^\infty dr\, \frac{e^{2\pi Tr(\Delta+n)}\cos kr}{(e^{4\pi Tr}-1)^{2\Delta-m-1}} = \frac{(-1)^{m-2\Delta}\Gamma(s)}{8\pi T}\left[\frac{\Gamma(z-s)}{\Gamma(z)} + \frac{\Gamma(\bar{z}-s)}{\Gamma(\bar{z})}\right], \tag{4.31}$$

and $s = 2 - 2\Delta + m$, and

$$z = 2 - \frac{3\Delta}{2} + m + \frac{n}{2} - \frac{ik}{4\pi T}. \tag{4.32}$$

In general $J_{n,m}(k)$ diverges, and the result on the right-hand side of eq. (4.31) should be understood as a limit of the regularized expression; after taking the limit, the (divergent) contact terms should be ignored.

As an example, consider the simplest case $\Delta = 1$. Taking the limit $s \to 0$ in eq. (4.31), we get

$$J_{n,0}(k) = -\frac{1}{8\pi T}\left[\psi(z) + \psi(\bar{z})\right]. \tag{4.33}$$

Therefore, we obtain

$$G^E(2\pi nT, k) = -\pi\, C_{\mathcal{O}}\left[\psi\left(\frac{1+n}{2} - \frac{ik}{4\pi T}\right) + \psi\left(\frac{1+n}{2} + \frac{ik}{4\pi T}\right)\right]. \tag{4.34}$$

Identifying the prefactors as $4\eta/l\epsilon \ln^2\epsilon = \pi C_{\mathcal{O}}$ one observes that the retarded Green's function (4.22) taken at $\omega = 2\pi i Tn$ equals (with a minus sign) to the expression given by eq. (4.34), as expected on general grounds in eq. (2.10), and therefore the field theory and gravity results are consistent.

For $\Delta = 2$, eqs. (4.30) and (4.31) give

$$G^E(2\pi nT, k) = \pi\, C_{\mathcal{O}} \frac{(2\pi Tn)^2 + k^2}{4}\left[\psi\left(1 + \frac{n}{2} - \frac{ik}{4\pi T}\right) + \psi\left(1 + \frac{n}{2} + \frac{ik}{4\pi T}\right)\right]. \tag{4.35}$$

Upon identification $\pi^3 C_{\mathcal{O}} = \eta/lT_RT_L$, (4.35) coincides with $-G^R(\omega = 2\pi Tn, k)$, where G^R was computed in eq. (4.19). One can repeat this procedure for any integer Δ. So we have checked that our prescription does give the correct retarded Green's functions in two-dimensional CFT.

4.3 Quasinormal modes and singularities of G^R

The quasinormal modes for black holes in asymptotically AdS spacetimes are defined as the solutions to the wave equation obeying the "incoming wave" boundary condition at the horizon and the vanishing Dirichlet condition at the boundary.[2] It was suggested in ref. [18] that the associated quasinormal frequencies are related to the process of thermalization in the dual strongly coupled CFT.[3] The inverse relaxation time is given by the imaginary part of the lowest quasinormal frequency, which has been computed numerically in ref. [18] for AdS black holes in various dimensions.

[2] For reviews and references on quasinormal modes in *asymptotically flat* spacetimes see [16, 17].

[3] Similar ideas appear in ref. [19]. Other works on quasinormal modes in asymptotically AdS spacetimes include refs. [20]–[24].

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Due to the simplicity of the BTZ background, the quasinormal modes in this case can be determined analytically [25, 26, 27]. It was first noticed in ref. [28] that the quasinormal frequencies found in ref. [25] correspond to the poles of the retarded Green's function describing the relaxation of the boundary CFT to thermal equilibrium, as in the linear response theory.

Our calculation of the retarded Green's function confirms this conjecture explicitly: the poles of G^R (eqs. (4.23)) are precisely the quasinormal frequencies of the BTZ black hole [25].

In fact, given the prescription (3.15), it is not difficult to see that this should be true in general. Indeed, suppose $\phi_k(z) = A(k)\phi_1(z) + B(k)\phi_2(z)$ is the (super)gravity field whose boundary value ϕ_0 couples to an operator $\mathcal{O}$ in the dual theory in Minkowski space, and suppose further that ϕ obeys the incoming wave boundary condition at the horizon. The normalized solution used to compute the retarded correlator has the form $\psi_k(z) = \phi_0(k)\phi_k(z)/\phi_k(\epsilon)$. Near the boundary $z = \epsilon \to 0$ one has $\psi_k(z) = A(k)\, z^{\Delta_-} + \cdots + B(k)\, z^{\Delta_+} + \cdots$. Then

$$G^R(k) \sim \psi^* \psi' = (\Delta_+ - \Delta_-) \frac{B(k)}{A(k)} \epsilon^{\Delta_+ - \Delta_- - 1} + \text{contact terms} + O(\epsilon)\,. \tag{4.36}$$

Zeros of $A(k)$ correspond to singularities of the retarded correlator. But $A(k) = 0$ is precisely the vanishing Dirichlet boundary condition at $z = 0$ which defines the quasinormal modes in asymptotically AdS spacetimes (recall that the boundary condition at the horizon is the same for the quasinormal modes and for the solution from which G^R is determined). Thus, quite generically, quasinormal frequencies are equal to the the resonance frequencies of the retarded Green's function of the operator $\mathcal{O}$. In the linear response theory, they characterize the time scale of the approach to thermal equilibrium in the boundary field theory perturbed by $\mathcal{O}$.

5. Thermal $\mathcal{N} = 4$ SYM theory and the Chern-Simons diffusion rate

We now consider the four-dimensional $\mathcal{N} = 4$ SYM theory at finite temperature. The thermodynamics of this theory has been studied in detail by using its gravity dual [29, 30]. Here, we compute a simple thermal correlation function by using the Minkowski AdS/CFT prescription.

The quantity we shall be interested in is the Chern-Simons diffusion rate,

$$\Gamma = \left(\frac{g^2_{\rm YM}}{8\pi^2} \right)^2 \int d^4x \, \langle \hat{\mathcal{O}}(x) \hat{\mathcal{O}}(0) \rangle \,, \tag{5.1}$$

where

$$\hat{\mathcal{O}} = \frac{1}{4} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \,. \tag{5.2}$$

This quantity is interesting because it has been intensively studied at weak coupling. In the standard model, Γ determines the rate of anomalous baryon number violation at high temperatures. Its knowledge is important for electroweak baryogenesis [31]. At weak coupling, the parametric behavior of Γ is [32]–[35]

$$\Gamma = \text{const} \cdot (g_{\rm YM}^2 N)^5 \ln \frac{1}{g_{\rm YM}^2 N}\, T^4 \,. \tag{5.3}$$

The overall numerical constant is of nonperturbative nature but can be found by a lattice simulation of a classical field theory [36, 37, 38]. No method has been developed to compute Γ at strong coupling.

In the original AdS/CFT correspondence, $\hat{\mathcal{O}}$ is coupled to the bulk axion, therefore the correlators of $\hat{\mathcal{O}}$ can be found from gravity. Indeed, if one finds the retarded propagator G^R for $\hat{\mathcal{O}}$, then, according to eq. (2.8),

$$\Gamma = -\left(\frac{g_{\rm YM}^2}{8\pi^2}\right)^2 \lim_{\omega\to 0} \frac{2T}{\omega}\, \mathrm{Im}\, G^R(\omega, \mathbf{0})\,. \tag{5.4}$$

Therefore, the task of computing Γ is reduced to the calculation of the retarded Green's function of $\hat{\mathcal{O}}$ at small momenta.

The coupling of the axion (the Ramond-Ramond scalar) C_0 to $\hat{\mathcal{O}}$ is determined from the three-brane Born-Infeld action [39, 40]. The action for C_0 is the same as for a minimally coupled scalar, eq. (3.6). It is convenient to use the coordinate u defined as

$$u = \frac{z^2}{z_H^2}\,. \tag{5.5}$$

The mode equation has the form

$$f_k'' - \frac{1+u^2}{u(1-u^2)} f_k' + \frac{\mathfrak{w}^2}{u(1-u^2)^2} f_k - \frac{\mathfrak{q}^2}{u(1-u^2)} f_k = 0\,, \tag{5.6}$$

where the following notations were introduced,

$$\mathfrak{w} = \frac{\omega}{2\pi T}\,, \qquad \mathfrak{q} = \frac{|\boldsymbol{k}|}{2\pi T}\,. \tag{5.7}$$

Equation (5.6) is the Heun differential equation [41] (second-order differential equation with four regular singular points, see [42, 43]). Global solutions to the Heun equation are unknown. However, for computing the Chern-Simons diffusion rate one needs to consider only the regime of low frequency and momentum, $\omega, |\boldsymbol{k}| \ll T$. In this regime, the solution representing the incoming wave at the horizon is given by

$$\phi_k(u) = (1-u)^{-i\mathfrak{w}/2} F_k(u)\,, \tag{5.8}$$

where $F_k(u)$ is regular at $u=1$ and can be written as a series,

$$F_k(u) = 1 - \left(\frac{i\mathfrak{w}}{2} + \mathfrak{q}^2\right) \ln\frac{1+u}{2} + O(\mathfrak{w}^2,\, \mathfrak{w}\mathfrak{q}^2,\, \mathfrak{q}^4)\,. \tag{5.9}$$

Computing the boundary term (3.12) near $u=0$ we obtain

$$\mathcal{F}(k,u) = -\frac{\pi^2 N^2 T^4}{8} \left[i\mathfrak{w} - \mathfrak{q}^2\left(1 - \frac{1}{u}\right)\right] + O(\mathfrak{w}^2,\, \mathfrak{w}\mathfrak{q}^2,\, \mathfrak{q}^4)\,. \tag{5.10}$$

When $\boldsymbol{k} = 0$, our prescription implies

$$G^{R/A}(\omega, \mathbf{0}) = \mp \frac{i\pi N^2 \omega T^3}{8} \left(1 + O\left(\frac{\omega}{T}\right)\right). \tag{5.11}$$

From eq. (5.4) one then finds

$$\Gamma = \frac{(g_{\mathrm{YM}}^2 N)^2}{256\pi^3} T^4 \,, \tag{5.12}$$

which is our final result for the Chern-Simons diffusion rate at large 't Hooft coupling $g_{\mathrm{YM}}^2 N \gg 1$. Note that (5.12) has the same large N behavior as the weak-coupling expression (5.3) in the 't Hooft limit (that is, of order N^0). The natural conjecture to make is that the Chern-Simon diffusion rate is proportional to T^4 with a proportionality coefficient dependent on the 't Hooft coupling,

$$\Gamma = f_\Gamma(g_{\mathrm{YM}}^2 N) T^4 \tag{5.13}$$

where $f_\Gamma(x) \sim x^5 \ln \frac{1}{x}$ for $x \ll 1$ and $f_\Gamma(x) = x^2/(256\pi^3)$ for $x \gg 1$.

Notice that if we considered, instead of $\hat{\mathcal{O}} = \frac{1}{4} F\tilde{F}$ the operators $\frac{1}{4}F^2$, the answer would be the same. The same result is also valid for spatial components of the stress-energy tensor if $\boldsymbol{k} = 0$, since the corresponding gravitational perturbation obeys the same equation [40]. For the latter case, we reproduce, from Kubo's formula, the value for the shear viscosity $\eta = \frac{\pi}{8} N^2 T^3$ that has been found in ref. [44].

The imaginary part of the retarded propagator $G(\omega, \boldsymbol{k})$ can also be computed for very high frequencies and momenta $(\omega, |\boldsymbol{k}| \gg T)$. This is done in appendix B.

6. Conclusions

We have formulated a prescription that allows one to compute the Minkowski retarded Green's function from gravity. We have demonstrated that the prescription works for zero-temperature gauge theory, and for finite-temperature two-dimensional CFT. By using the prescription, we found the Chern-Simons diffusion rate for the $\mathcal{N} = 4$ SYM theory at finite-temperature. The prescription allows one to compute many other quantities which cannot be computed from euclidean space. Two examples, the shear viscosity and the R-charge diffusion rate in thermal $\mathcal{N} = 4$ SYM theory, are presented in ref. [10].

The prescription given here does not allow one to find higher-point correlation function, and hence it cannot be the final word on Minkowski gauge theory/gravity duality. Hopefully, it can be incorporated in some future more general scheme.

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A. Zero-temperature correlation functions in euclidean and Minkowski AdS/CFT

A.1 Euclidean signature

To compute the euclidean two-point function of a CFT operator $\mathcal{O}$ one uses the AdS/CFT equivalence

$$\langle e^{\int_{\partial M} \phi_0 \mathcal{O}} \rangle = e^{-S_E[\phi]} \,, \tag{A.1}$$

where $S_E[\phi]$ is the classical (super)gravity action on M and ϕ_0 is the boundary value of the bulk field ϕ. At zero temperature $M = \mathrm{AdS}_5 \times \mathrm{S}^5$. These calculations were first done in ref. [2] and then appeared in the literature many times. We repeat them here to have an explicit comparison with the lorentzian version. The (euclidean) AdS_5 part of the metric is given by

$$ds^2_{d+1} = \frac{R^2}{z^2}(dz^2 + d\boldsymbol{x}^2) \,, \tag{A.2}$$

where $\boldsymbol{x}$ are the coordinates on R^4. The action of the massive dilaton fluctuation ϕ is

$$S_E = \frac{\pi^3 R^8}{4\kappa_{10}^2} \int dz\, d^4x\, z^{-3} \left[(\partial_z \phi)^2 + (\partial_i \phi)^2 + \frac{m^2 R^2}{z^2} \phi^2 \right] . \tag{A.3}$$

Using the Fourier representation,

$$\phi(z,x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx} f_k(z) \phi_0(k) \,, \tag{A.4}$$

we obtain

$$S_E = \frac{\pi^3 R^8}{4\kappa_{10}^2} \int_\epsilon^\infty dz \int \frac{d^4k d^4k'}{(2\pi)^4}\, \frac{\delta(k+k')}{z^3} \left[\partial_z f_k \partial_z f_{k'} - kk' f_k f_{k'} + \frac{m^2 R^2}{z^2} f_k f_{k'} \right] \phi_0(k) \phi_0(k') \,. \tag{A.5}$$

The equation of motion for $f_k(z)$ reads

$$f_k'' - \frac{3}{z} f_k' - \left(k^2 + \frac{m^2 R^2}{z^2} \right) f_k = 0 \,. \tag{A.6}$$

It has a general solution

$$\phi_k(z) = A z^2 I_\nu(kz) + B z^2 I_{-\nu}(kz) \,, \tag{A.7}$$

where $\nu = \sqrt{4 + m^2 R^2}$ (or equivalently one can consider I_ν and K_ν as two independent solutions). We also use the standard notation $\Delta = \nu + 2$ for the conformal weight of the operator $\mathcal{O}$. The solution regular at $z = \infty$ and equal 1 at $z = \epsilon$ is given by

$$f_k(z) = \frac{z^2 K_\nu(kz)}{\epsilon^2 K_\nu(k\epsilon)} \,. \tag{A.8}$$

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On shell, the action reduces to the boundary terms,

$$\begin{aligned} S_E &= \frac{\pi^3 R^8}{4\kappa_{10}^2} \int \frac{d^4k d^4k'}{(2\pi)^8} (2\pi)^4 \delta^4(k+k') \phi_0(k) \phi_0(k') \frac{f_{k'}(z)\partial_z f_k(z)}{z^3} \bigg|_\epsilon^\infty \\ &\equiv \int \frac{d^4k d^4k'}{(2\pi)^8} \phi_0(k)\phi_0(k') \mathcal{F}(z,k,k') \Big|_\epsilon^\infty . \end{aligned} \tag{A.9}$$

We note that $\mathcal{F}(z,k,k')$ is a real function, and $\mathcal{F}(\infty,k,k')=0$. The two-point function is given by

$$\begin{aligned} \langle \hat{\mathcal{O}}(k)\hat{\mathcal{O}}(k')\rangle &= Z^{-1} \frac{\delta^2 Z[\phi_0]}{\delta\phi_0(k)\delta\phi_0(k')} \bigg|_{\phi_0=0} = -2\mathcal{F}(z,k,k') \Big|_\epsilon^\infty \\ &= -(2\pi)^4 \delta^4(k+k') \frac{\pi^3 R^8}{2\kappa_{10}^2} \frac{f_{k'}(z)\partial_z f_k(z)}{z^3} \bigg|_\epsilon^\infty . \end{aligned} \tag{A.10}$$

Direct calculation from eq. (A.8) gives

$$\langle \hat{\mathcal{O}}(k)\hat{\mathcal{O}}(k')\rangle = -\frac{\pi^3 R^8}{2\kappa_{10}^2} \epsilon^{2(\Delta-d)} (2\pi)^4 \delta^4(k+k') k^{2\nu} 2^{1-2\nu} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} + \cdots \tag{A.11}$$

where dots denote terms analytic in k and/or those vanishing in the $\epsilon \to 0$ limit. Substituting $\kappa_{10} = 2\pi^{5/2} R^4/N$ [29] we get

$$\langle \hat{\mathcal{O}}(k)\hat{\mathcal{O}}(k')\rangle = -\frac{N^2}{8\pi^2} \epsilon^{2(\Delta-4)} (2\pi)^4 \delta^4(k+k') \frac{k^{2\Delta-4}\Gamma(3-\Delta)}{2^{2\Delta-5}\Gamma(\Delta-2)} . \tag{A.12}$$

For integer Δ, the propagator is given by

$$\langle \hat{\mathcal{O}}(k)\hat{\mathcal{O}}(k')\rangle = -\frac{(-1)^\Delta}{(\Delta-3)!^2} \frac{N^2}{8\pi^2} \epsilon^{2(\Delta-4)} (2\pi)^4 \delta^4(k+k') \frac{k^{2\Delta-4}}{2^{2\Delta-5}} \ln k^2 , \quad \Delta = 2,3,4,\ldots \tag{A.13}$$

where one sets $(-1)! = 1$. Specializing further to the massless case ($\Delta = 4$), we have

$$\langle \hat{\mathcal{O}}(k)\hat{\mathcal{O}}(k')\rangle = -\frac{N^2}{64\pi^2} (2\pi)^4 \delta^4(k+k') k^4 \ln k^2 . \tag{A.14}$$

A.2 Lorentzian signature

In the AdS$_5$ background the action describing the dilaton fluctuation reads

$$S = -\frac{\pi^3 R^8}{4\kappa_{10}^2} \int d^4x \int_\epsilon^\infty dz \, \frac{dz}{z^3} \left[(\partial_z\phi)^2 + \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{m^2R^2}{z^2}\phi^2 \right] , \tag{A.15}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and R is the AdS_5 radius. Writing the classical solution as

$$\phi(z,t,\boldsymbol{x}) = \int \frac{d^4k}{(2\pi)^4} \, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} f_k(z)\phi_0(k) , \tag{A.16}$$

the action reduces to the boundary term

$$S = \int \frac{d^4k}{(2\pi)^4} \Big[\mathcal{F}(k,\infty) - \mathcal{F}(k,\epsilon)\Big] \phi_0(-k)\phi_0(k)\,, \tag{A.17}$$

where

$$\mathcal{F}(k,z) = -\frac{\pi^3 R^8}{4\kappa_{10}^2} \frac{f_{-k}\partial_z f_k(z)}{z^3}\,. \tag{A.18}$$

The mode function $f_k(z)$ obeys the same equation (A.6) as in the euclidean case. However, now the form of the solution depends on whether k is spacelike or timelike.

A.2.1 Spacelike momenta, $k^2 > 0$

For spacelike momenta, the solution is real and the calculation is identical to that of the euclidean case, except for the additional minus sign coming from the difference in definitions of euclidean and Minkowski actions. The result is thus given by

$$G^R(\omega, \boldsymbol{k}) = \frac{N^2 (k^2)^{\Delta-2} \Gamma(3-\Delta) \epsilon^{2(\Delta-4)}}{8\pi^2 2^{2\Delta-5} \Gamma(\Delta-2)}\,. \tag{A.19}$$

A.2.2 Timelike momenta, $k^2 < 0$

Introducing $q = \sqrt{-k^2} = |k|$, the solution to (A.6) satisfying the "incoming wave" boundary condition at $z = \infty$ and normalized to 1 at $z = \epsilon$ is

$$\phi_k(z) = \begin{cases} \dfrac{z^2 H_\nu^{(1)}(qz)}{\epsilon^2 H_\nu^{(1)}(q\epsilon)}\,, & \omega > 0\,, \\[2ex] \dfrac{z^2 H_\nu^{(2)}(qz)}{\epsilon^2 H_\nu^{(2)}(q\epsilon)}\,, & \omega < 0\,. \end{cases} \tag{A.20}$$

Here $\nu = \sqrt{4 + m^2 R^2}$. Using the expansion

$$z^2 H_\nu^{(1,2)}(z) = \pm \frac{1}{\sin \nu\pi} \left[-\frac{2^\nu z^{2-\nu}}{\Gamma(1-\nu)} \left(1 + O(z^2)\right) + e^{\mp \nu\pi i} \frac{z^{\nu+2}}{2^\nu \Gamma(\nu+1)} \left(1 + O(z^2)\right) \right], \tag{A.21}$$

and the prescription (3.15), one obtains the retarded Green's function for timelike momenta

$$G^R(k) = \frac{N^2 (q^2)^{\Delta-2} \Gamma(3-\Delta) \epsilon^{2(\Delta-4)}}{8\pi^2 2^{2\Delta-5} \Gamma(\Delta-2)} \left[\cos \pi\Delta - i \sin(\pi\Delta) \operatorname{sgn} \omega\right]. \tag{A.22}$$

Here $\Delta = 2 + \nu \neq 2, 3, \ldots$. For integer values of Δ, by taking the appropriate limit in eq. (A.22), one has

$$G^R(k) = \frac{N^2 \epsilon^{2(\Delta-4)}}{8\pi^2 (\Delta-3)!^2 2^{2\Delta-5}} (q^2)^{\Delta-2} \left[\ln q^2 - i\pi \operatorname{sgn} \omega\right], \qquad \Delta = 2, 3, 4, \ldots \tag{A.23}$$

where one sets $(-1)! = 1$.

Combining the results for the time- and spacelike momenta, one can write

$$\operatorname{Re} G^R(k) = \begin{cases} \dfrac{N^2|k^2|^{\Delta-2}\Gamma(3-\Delta)\epsilon^{2(\Delta-4)}}{8\pi^2 2^{2\Delta-5}\Gamma(\Delta-2)}\,, & k^2>0\,, \\ \dfrac{N^2|k^2|^{\Delta-2}\Gamma(3-\Delta)\epsilon^{2(\Delta-4)}}{8\pi^2 2^{2\Delta-5}\Gamma(\Delta-2)}\cos\pi\Delta\,, & k^2<0\,, \end{cases} \tag{A.24a}$$

$$\operatorname{Im} G^R(k) = -\frac{N^2|k^2|^{\Delta-2}\Gamma(3-\Delta)\epsilon^{2(\Delta-4)}}{8\pi^2 2^{2\Delta-5}\Gamma(\Delta-2)}\sin(\pi\Delta)\,\theta(-k^2)\operatorname{sgn}\omega\,. \tag{A.24b}$$

For the integer values of Δ, we have, correspondingly,

$$\operatorname{Re} G^R(k) = \frac{N^2\epsilon^{2(\Delta-4)}}{8\pi^2(\Delta-3)!^2 2^{2\Delta-5}}|k^2|^{\Delta-2}\ln|k^2|\,, \qquad \Delta=2,3,4,\ldots \tag{A.25a}$$

$$\operatorname{Im} G^R(k) = -\frac{N^2\epsilon^{2(\Delta-4)}}{8\pi(\Delta-3)!^2 2^{2\Delta-5}}|k^2|^{\Delta-2}\,\theta(-k^2)\ \operatorname{sgn}\omega\,, \qquad \Delta=2,3,4,\ldots \tag{A.25b}$$

In the massless case ($\Delta=4$), we get

$$G^R(k) = \frac{N^2k^4}{64\pi^2}(\ln|k^2| - i\pi\,\theta(-k^2)\operatorname{sgn}\omega)\,. \tag{A.26}$$

B. Retarded Green's function at high momenta

In this appendix we consider the retarded Green's function of $\hat{\mathcal{O}}=\frac{1}{4}F^2$ (the same formulas apply for $\hat{\mathcal{O}}=\frac{1}{4}F\tilde{F}$) in the regime $\omega,|\boldsymbol{k}|\gg T$. Since the temperature now is much smaller than the momentum scales, one should expect the result to be close to the zero-temperature result (3.20), with the temperature providing only a small correction. This can be shown explicitly by using the Langer-Olver method [4, 45]. However, one qualitative effect occurs at finite temperature: $G^R(k)$ acquires a small imaginary part for spacelike k's (recall that at zero temperature, $G^R(k)$ is real if $k^2>0$). This happens for the same reason as the one causing the Landau damping in plasma: excitations in the plasma can absorb a spacelike momentum. We shall estimate $\operatorname{Im} G^R(k)$ for large spacelike k's.

By a change of variables from $f_k(u)$ to $W(u)=u^{-1/2}\sqrt{u^2-1}\,f_k(u)$, the mode equation (5.6) can be written in the form of the Schrödinger equation,

$$W''(u) = \big(\mathfrak{w}^2F(u)+G(u)\big)\,W(u)\,, \tag{B.1}$$

where

$$F(u) = -\frac{1-s^2(1-u^2)}{u(1-u^2)^2}\,, \qquad G(u) = -\frac{u^4+6u^2-3}{4u^2(1-u^2)^2}\,, \tag{B.2}$$

and $s=|\boldsymbol{k}|/\omega$. In the regime, $\mathfrak{w}=(\omega R)^2/2\rho_0=\omega/2\pi T\gg 1$, $s\sim 1$, the term proportional to $\mathfrak{w}^2$ dominates in the potential, and the solution can be found by the WKB approximation.

For $s>1$ (corresponding to timelike k's), the potential in the Schrödinger equation (B.1) is positive for $u<u_0=\sqrt{1-1/s^2}$ and negative for $u>u_0$. The solution to eq. (B.1), thus, decays exponentially in the interval $(0,u_0)$ and oscillates in the interval

$(u_0, 1)$. Physically, the particle has to tunnel from $u = 0$ to $u = u_0$ before it can reach the horizon $u = 1$. The imaginary part of G^R is proportional to the tunneling probability, which is

$$\operatorname{Im} G^R(\omega, \boldsymbol{k}) \sim \exp\left(-2\mathfrak{w}\int_0^{u_0} du\, \sqrt{F(u)}\right). \tag{B.3}$$

For example, if $\omega \ll |\boldsymbol{k}|$, then

$$\operatorname{Im} G^R(\omega, \boldsymbol{k}) \sim e^{-a|\boldsymbol{k}|/T}, \qquad \omega \ll |\boldsymbol{k}|, \tag{B.4}$$

where

$$a = \frac{2\Gamma\left(5/4\right)}{\sqrt{\pi}\Gamma\left(3/4\right)} \approx 0.835. \tag{B.5}$$

It is not immediately clear how to interpret this value of a. In a weakly coupled relativistic plasma, an external perturbation with spatial momentum $(0, |\boldsymbol{k}|)$ can be absorbed by an particle with momentum $(|\boldsymbol{k}|/2, -|\boldsymbol{k}|/2)$ so that its momentum becomes $(|\boldsymbol{k}|/2, |\boldsymbol{k}|/2)$ and the particle remains on shell. The probability of such a process is suppressed by the Boltzmann factor of the particle, i.e., $e^{-|\boldsymbol{k}|/(2T)}$. Since $a > 1/2$, there is some additional suppression in the strongly coupled CFT.

The formulas presented above are confirmed by the more sophisticated Langer-Olver method, which is the WKB approximation for potentials with regular singular points (for more information and references see refs. [4, 45]).

References

[1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] G. Policastro and A. Starinets, *On the absorption by near-extremal black branes*, *Nucl. Phys.* **B 610** (2001) 117 [hep-th/0104065].

[5] V. Balasubramanian, P. Kraus, A.E. Lawrence and S.P. Trivedi, *Holographic probes of anti-de Sitter space-times*, *Phys. Rev.* **D 59** (1999) 104021 [hep-th/9808017].

[6] V. Balasubramanian, S.B. Giddings and A.E. Lawrence, *What do CFTs tell us about anti-de Sitter spacetimes?*, *J. High Energy Phys.* **03** (1999) 001 [hep-th/9902052].

[7] V. Balasubramanian, P. Kraus and A.E. Lawrence, *Bulk versus boundary dynamics in anti-de Sitter spacetime*, *Phys. Rev.* **D 59** (1999) 046003 [hep-th/9805171].

[8] U.H. Danielsson, E. Keski-Vakkuri and M. Kruczenski, *Vacua, propagators and holographic probes in AdS/CFT*, *J. High Energy Phys.* **01** (1999) 002 [hep-th/9812007].

[9] S. Ryang, *The Hadamard function and the Feynman propagator in the AdS/CFT correspondence*, *Phys. Lett.* **B 469** (1999) 87 [hep-th/9909186].

JHEP09(2002)042

[10] G. Policastro, D.T. Son and A.O. Starinets, *From AdS/CFT correspondence to hydrodynamics*, hep-th/0205052.

[11] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099].

[12] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys. Rev.* **D 48** (1993) 1506 [gr-qc/9302012].

[13] S.S. Gubser, *Absorption of photons and fermions by black holes in four dimensions*, *Phys. Rev.* **D 56** (1997) 7854 [hep-th/9706100].

[14] J.L. Cardy, *Conformal invariance and universality in finite size scaling, J. Phys.* **A 17** (1984) L385.

[15] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integraly i ryady. Elementarnye funktsii*, Nauka, Moscow 1981. English translation: *Integrals and series. Vol. 1: elementary functions*, Gordon & Breach Science Publishers, New York 1986.

[16] V. P. Frolov and I. D. Novikov, *Black hole physics: basic concepts and new developments*, Kluwer Academic Publishers, Dordrecht 1998.

[17] K.D. Kokkotas and B.G. Schmidt, *Quasi-normal modes of stars and black holes*, *Living Rev. Rel.* **2** (1999) 2 [gr-qc/9909058].

[18] G.T. Horowitz and V.E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium, Phys. Rev.* **D 62** (2000) 024027 [hep-th/9909056].

[19] S. Kalyana Rama and B. Sathiapalan, *On the role of chaos in the AdS/CFT connection*, *Mod. Phys. Lett.* **A 14** (1999) 2635 [hep-th/9905219].

[20] J.S.F. Chan and R.B. Mann, *Scalar wave falloff in asymptotically anti-de Sitter backgrounds*, *Phys. Rev.* **D 55** (1997) 7546 [gr-qc/9612026].

[21] B. Wang, C.-Y. Lin and E. Abdalla, *Quasinormal modes of Reissner-Nordstroem anti-de Sitter black holes*, *Phys. Lett.* **B 481** (2000) 79 [hep-th/0003295].

[22] V. Cardoso and J.P.S. Lemos, *Quasi-normal modes of Schwarzschild anti-de sitter black holes: electromagnetic and gravitational perturbations*, *Phys. Rev.* **D 64** (2001) 084017 [gr-qc/0105103].

[23] B. Wang, E. Abdalla and R.B. Mann, *Scalar wave propagation in topological black hole backgrounds*, *Phys. Rev.* **D 65** (2002) 084006 [hep-th/0107243].

[24] I.G. Moss and J.P. Norman, *Gravitational quasinormal modes for anti-de sitter black holes*, *Class. and Quant. Grav.* **19** (2002) 2323 [gr-qc/0201016].

[25] D. Birmingham, *Choptuik scaling and quasinormal modes in the AdS/CFT correspondence*, *Phys. Rev.* **D 64** (2001) 064024 [hep-th/0101194].

[26] T.R. Govindarajan and V. Suneeta, *Quasi-normal modes of AdS black holes: a superpotential approach*, *Class. and Quant. Grav.* **18** (2001) 265 [gr-qc/0007084].

[27] V. Cardoso and J.P.S. Lemos, *Scalar, electromagnetic and Weyl perturbations of BTZ black holes: quasi normal modes*, *Phys. Rev.* **D 63** (2001) 124015 [gr-qc/0101052].

[28] D. Birmingham, I. Sachs and S.N. Solodukhin, *Conformal field theory interpretation of black hole quasi-normal modes*, *Phys. Rev. Lett.* **88** (2002) 151301 [hep-th/0112055].

JHEP09(2002)042

[29] S.S. Gubser, I.R. Klebanov and A.W. Peet, *Entropy and temperature of black 3-branes*, *Phys. Rev.* **D 54** (1996) 3915 [hep-th/9602135].

[30] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *Coupling constant dependence in the thermodynamics of $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 534** (1998) 202 [hep-th/9805156].

[31] V.A. Rubakov and M.E. Shaposhnikov, *Electroweak baryon number non-conservation in the early universe and in high-energy collisions*, *Usp. Fiz. Nauk* **166** (1996) 493–537 [hep-ph/9603208].

[32] P. Arnold, D. Son and L.G. Yaffe, *The hot baryon violation rate is $O(\alpha_w^5 T^4)$*, *Phys. Rev.* **D 55** (1997) 6264 [hep-ph/9609481].

[33] P. Huet and D.T. Son, *Long range physics in a hot non-abelian plasma*, *Phys. Lett.* **B 393** (1997) 94 [hep-ph/9610259].

[34] D. Bödeker, *On the effective dynamics of soft non-abelian gauge fields at finite temperature*, *Phys. Lett.* **B 426** (1998) 351 [hep-ph/9801430].

[35] G.D. Moore, *Do we understand the sphaleron rate?*, hep-ph/0009161.

[36] C.R. Hu and B. Müller, *Classical lattice gauge field with hard thermal loops*, *Phys. Lett.* **B 409** (1997) 377 [hep-ph/9611292].

[37] G.D. Moore, C.-R. Hu and B. Müller, *Chern-Simons number diffusion with hard thermal loops*, *Phys. Rev.* **D 58** (1998) 045001 [hep-ph/9710436].

[38] G.D. Moore, *The sphaleron rate: Böedeker's leading log*, *Nucl. Phys.* **B 568** (2000) 367 [hep-ph/9810313].

[39] I.R. Klebanov, *World-volume approach to absorption by non-dilatonic branes*, *Nucl. Phys.* **B 496** (1997) 231 [hep-th/9702076].

[40] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *String theory and classical absorption by three-branes*, *Nucl. Phys.* **B 499** (1997) 217 [hep-th/9703040].

[41] K. Heun, *Zur theorie der Riemann's sehen functionen zweiter ordnung mit vier verzweigungspunkten*, *Math. Ann.* **33** (1889) 61.

[42] *Higher transcendental functions, vol. III*, A. Erdelyi ed., McGraw Hill, New York 1955.

[43] *Heun's differential equation*, A. Ronveaux, F. Arscott eds., Oxford University Press, Oxford 1995.

[44] G. Policastro, D.T. Son and A.O. Starinets, *The shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **87** (2001) 081601 [hep-th/0104066].

[45] F.W.J. Olver, *Asymptotics and special functions*, A K Peters, Wellesley 1997.

JHEP09(2002)042